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Alternative Problems and Symmetry

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The results of Hale [1, 2] for systems which have the property E are extended to general nonlinear equations and to general (finite) symmetry groups.

1. INTRODUCTION

In this paper we show how the additional symmetry of a nonlinear equation helps to reduce the determining equations for some of the solutions. To this purpose, we generalize the theory for periodic solutions of systems, which have the so-called property E, as introduced by Hale [1, 2]. Here we consider general nonlinear equations and general (finite) symmetry groups. The main point is that one should formulate the alternative problem in a symmetric way, in order to preserve the effect of the symmetries in the determining equations. In a last section we indicate some applications of the method.

2. AN ALTERNATIVE PROBLEM

We consider Banach spaces X and Z, and the equation:

$$Lx = Nx \tag{2.1}$$

where

(i) $L\colon \operatorname{dom} L \subset X \to Z$ is linear,

(ii) $N\colon X \to Z$ is not necessarily linear.

Solutions of (2.1) are elements of $\operatorname{dom} L$, satisfying (2.1). Now we assume the following hypothesis:

(H1) There exist projections $P\colon X \to X$ and $Q\colon Z \to Z$ such that:

$$\ker L = \operatorname{Im} P \qquad \text{and} \qquad \operatorname{Im} L = \ker Q. \tag{2.2}$$

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Then $L_P = L|_{\mathrm{dom} L \cap \ker P}$ is an algebraic isomorphism between $\mathrm{dom}\, L \cap \ker P$ and $\mathrm{Im}\, L \subset Z$, and has an inverse $K_P = L_P^{-1}$, defined on $\mathrm{Im}\, L$. We have:

$$LK_P = I \qquad \text{on} \qquad \mathrm{Im}\, L, \tag{2.3}$$

$$K_P L = I - P \qquad \text{on} \qquad \mathrm{dom}\, L, \tag{2.4}$$

which also implies:

$$PK_P = 0 \qquad \text{on} \qquad \mathrm{Im}\, L.$$

THEOREM 2.1. *Under Hypothesis* (H1), (2.1) *is equivalent to the following system of equations:*

$$x = Px + K_P(I - Q)\, Nx, \tag{2.5a}$$

$$QNx = 0. \tag{2.5b}$$

For the proof, see, e.g., Hale [2] and Bancroft *et al.* [3]. Under some suitable conditions on N (see, e.g., [3]), one may solve (2.5a), when $y = Px$ is given in some domain Ω of $\mathrm{Im}\, P$. This means that the equation

$$x = y + K_P(I - Q)\, Nx, \qquad y \in \Omega \subset \mathrm{Im}\, P \tag{2.6}$$

has a unique solution $\bar{x}(y)$. Then (2.5b) becomes an equation in y alone, the so-called determining equation

$$QN\bar{x}(y) = 0, \qquad y \in \Omega \subset \mathrm{Im}\, P. \tag{2.7}$$

This equation is, in principle, simpler than the original one (2.1), as the space of y, namely, $\mathrm{Im}\, P$, is smaller than X.

3. Equations with Additional Symmetry

Now we assume that (2.1) has some additional symmetry properties. We make the following hypothesis concerning L:

(H2) There exists a finite group of linear, continuous operators in X:

$$\mathscr{G} = \{S_i\colon X \to X,\ i = 1,\dots, n\}$$

and a homomorphous group of linear continuous operators in Z:

$$\mathscr{G}^* = \{S_i^*\colon Z \to Z,\ i = 1,\dots, n\}$$

such that

(i) $\mathrm{dom}\, L$ is symmetric for the operators S_i:

$$S_i\colon \mathrm{dom}\, L \to \mathrm{dom}\, L \qquad (i = 1,\dots, n);$$

(ii) $LS_i = S_i^* L \qquad (i = 1,\dots, n).$ (3.1)

We remark that, although some of the $S_i{}^*$ may coincide, we will attach to them the same indices $i = 1,..., n$ as for the S_i.

We will assume an analogous hypothesis concerning N:

(H3) For the groups $\mathscr{G}$ and $\mathscr{G}^*$ from (H2), we have:

$$NS_i = S_i{}^*N \qquad (i = 1,..., n). \tag{3.2}$$

THEOREM 3.1. *Suppose* (H2) *and* (H3) *are satisfied. Let* $x^* \in \operatorname{dom} L$ *be a solution of* (2.1). *Then, for each* $i = 1,..., n$,

$$x_i{}^* = S_i x^* \tag{3.3}$$

is also a solution of (2.1).

The proof is immediate.

Now we want to study the following question: Assume (H1), (H2), and (H3). How does the symmetry property for the solutions, given by Theorem 3.1, appear in the alternative problem (2.7)?

4. SYMMETRIC PROJECTIONS

When considering alternative problems, one has, first of all, to consider projection operators P and Q, as given by Hypothesis (H1). The conditions imposed on P and Q depend on L, but do not uniquely determine P and Q. In order to maintain symmetry properties in the alternative problem, one must choose P and Q in a particular way, as will be seen from the following. As this part of the theory only depends on L, we will assume (H1) and (H2) throughout this section.

LEMMA 4.1. *The restriction* $S_i|_{\ker L}$ *of* S_i *to* $\ker L$ *is a continuous automorphism of* $\ker L$.

The restriction $S_i{}^*|_{\operatorname{Im} L}$ *of* $S_i{}^*$ *to* $\operatorname{Im} L$ *is a continuous automorphism of* $\operatorname{Im} L$.

Proof. We have:

$$Lx = 0 \Rightarrow S_i{}^*Lx = 0 \Rightarrow L(S_i x) = 0 \Rightarrow S_i x \in \ker L,$$

while, by the group properties, S_i has an inverse with the same properties.

Further, when $z \in \operatorname{Im} L$, there exists an $x \in \operatorname{dom} L$, such that $Lx = z$. This implies $S_i x \in \operatorname{dom} L$, and $S_i{}^*z = S_i{}^*Lx = L(S_i x) \in \operatorname{Im} L$.

LEMMA 4.2. *Let* $P: X \to X$ *be a projection, with* $\ker L = \operatorname{Im} P$. *Then the projections:*

$$P_i = S_i^{-1} P S_i \qquad (i = 1,..., n) \tag{4.1}$$

also satisfy:

$$\ker L = \operatorname{Im} P_i \qquad (i = 1,..., n). \tag{4.2}$$

Proof. P_i is continuous, linear, and

$$P_i^2 = S_i^{-1}P^2S_i = S_i^{-1}PS_i = P_i\,.$$

Further:

$$\begin{aligned}\operatorname{Im} P_i &= P_i(X) = S_i^{-1}PS_i(X) = S_i^{-1}P(X) = S_i^{-1}(\operatorname{Im} P)\\ &= S_i^{-1}(\ker L) = \ker L.\end{aligned}$$

LEMMA 4.3. *Let $Q: Z \to Z$ be a projection, with $\ker Q = \operatorname{Im} L$. Then the projections:*

$$Q_i = S_i^{*-1}QS_i^* \qquad (i = 1,..., n) \tag{4.3}$$

also satisfy

$$\ker Q_i = \operatorname{Im} L. \tag{4.4}$$

Proof. Let $y \in \ker Q_i$. Then $QS_i^*y = 0$, and so $S_i^*y \in \ker Q = \operatorname{Im} L$. This implies $y \in \operatorname{Im} L$.

Inversely, let $y \in \operatorname{Im} L$. Then $S_i^*y \in \operatorname{Im} L$, so $QS_i^*y = 0$ and $Q_iy = 0$.

We also need the following lemma's, the proof of which is classical.

LEMMA 4.4. *Let X be a Banach space, and $P_i: X \to X$ $(i = 1,..., n)$ be projections, with*

$$\operatorname{Im} P_1 = \operatorname{Im} P_2 = \cdots = \operatorname{Im} P_n\,. \tag{4.5}$$

Then, when $\sum_i a_i = 1$,

$$P = \sum_i a_iP_i \tag{4.6}$$

is also a projection, with $\operatorname{Im} P = \operatorname{Im} P_i$.

LEMMA 4.5. *Suppose the conditions of Lemma* 4.4, *with* (4.5) *replaced by*

$$\ker P_1 = \ker P_2 = \cdots = \ker P_n\,. \tag{4.7}$$

Then P, given by (4.6), is a projection, now with $\ker P = \ker P_i$.

THEOREM 4.1. *Assume* (H1) *and* (H2). *Then the projections:*

$$P_S = \frac{1}{n}\sum_i S_i^{-1}PS_i \tag{4.8}$$

and

$$Q_S = \frac{1}{n} \sum_i S_i^{*-1} Q S_i^* \tag{4.9}$$

also satisfy the conditions of (H1).

The proof is immediate from the foregoing lemma's.

THEOREM 4.2. *Under the conditions of Theorem* 4.1, *we have*

$$S_i P_S = P_S S_i \qquad (i = 1, \dots, n) \tag{4.10}$$

and

$$S_i^* Q_S = Q_S S_i^* \qquad (i = 1, \dots, n). \tag{4.11}$$

Proof.

$$S_i P_S = \frac{1}{n} \sum_j (S_j S_i^{-1})^{-1} P (S_j S_i^{-1}) S_i = \frac{1}{n} \sum_k S_k^{-1} P S_k S_i = P_S S_i,$$

where use is made of the rearrangement theorem for group elements. An analogous proof holds for (4.11).

LEMMA 4.6. (i) *The restriction* $S_i|_{\ker P_S}$ *of* S_i *to* $\ker P_S$ *is a continuous automorphism of* $\ker P_S$.

(ii) *The restriction* $S_i^*|_{\operatorname{Im} Q_S}$ *of* S_i^* *to* $\operatorname{Im} Q_S$ *is a continuous automorphism of* $\operatorname{Im} Q_S$.

Proof. (i) Let $x \in \ker P_S$. Then $P_S(S_i x) = S_i(P_S x) = 0$.

(ii) Let $y \in \operatorname{Im} Q_S$. Then $y = Q_S y$, and

$$S_i^* y = S_i^* Q_S y = Q_S(S_i^* y).$$

THEOREM 4.3. *Under the conditions of Theorem* 4.1, *we have on* $\operatorname{Im} L$:

$$K_{P_S} S_i^* = S_i K_{P_S} \qquad (i = 1, \dots, n). \tag{4.12}$$

Proof. Both applications are defined from $\operatorname{Im} L$ into $\operatorname{dom} L \cap \ker P_S$. We have also (2.3) and (2.4), with P and K_P replaced by P_S and K_{P_S}. Then take $z \in \operatorname{Im} L$. There exists $x \in \operatorname{dom} L$ such that $z = Lx$, and

$$\begin{aligned} K_{P_S} S_i^* z &= K_{P_S} S_i^* L x = K_{P_S} L S_i x = (I - P_S) S_i x \\ &= S_i (I - P_S) x = S_i K_{P_S} L x = S_i K_{P_S} z, \qquad \forall z \in \operatorname{Im} L. \end{aligned}$$

Conclusion. It follows from the foregoing that, when (H1) and (H2) are satisfied, it is always possible to choose P and Q in (H1) so that:

$$P S_i = S_i P, \qquad Q S_i^* = S_i^* Q \qquad (i = 1, \dots, n). \tag{4.13}$$

Then we have also:

$$K_P S_i^* = S_i K_P \qquad \text{on} \qquad \operatorname{Im} L. \tag{4.14}$$

In what follows we will always assume that P and Q are chosen in this symmetric way.

5. Symmetry of the Alternative Problem

In this section we assume (H1), (H2), and (H3). Then (2.1) is equivalent to (2.5). As all operators appearing in (2.5) commute in a definite way with the symmetry operators S_i and S_i^*, the conclusion of Theorem 3.1 can now be checked immediately on (2.5) itself, without any reference to the original equation (2.1).

THEOREM 5.1. *Let $x^* \in X$ be a solution of* (2.5). *Then $x_i^* = S_i x^*$ is also a solution of these equations.*

We now consider the equation

$$x = y + K_P(I - Q)\, Nx \tag{5.1}$$

for $y \in \operatorname{Im} P = \ker L$. About the solutions of this equation we can state the following lemma.

LEMMA 5.1. *Let, for $y \in \ker L$, $\bar{x} \in \operatorname{dom} L$ be a solution of* (5.1). *Then $\bar{x}_i = S_i \bar{x}$ $(i = 1,..., n)$ is a solution of the analogous equation*

$$x = S_i y + K_P(I - Q)\, Nx. \tag{5.2}$$

The proof is immediate.

LEMMA 5.2. *Let*

$$\Omega = \{y \in \ker L\colon (5.1) \textit{ has a unique solution } \bar{x} \in \operatorname{dom} L\}. \tag{5.3}$$

Then Ω is symmetric for the operators S_i:

$$S_i(\Omega) = \Omega \qquad (i = 1,..., n).$$

Proof. Let $y \in \Omega$, and $\bar{x}$ the unique solution of (5.1). Then $\bar{x}_i$ is a solution of (5.2), which must be unique, for otherwise (5.1) would have more than one solution. So $S_i y \in \Omega$.

If $y \in \Omega$, we denote by $\bar{x}(y)$ the unique solution of (5.1). This defines an application:

$$\bar{x}\colon \Omega \to \operatorname{dom} L, \qquad y \mapsto \bar{x}(y).$$

THEOREM 5.2. *With the foregoing definitions, we have on Ω:*

$$S_i\bar{x} = \bar{x}S_i \qquad (i = 1,..., n). \tag{5.4}$$

Proof. As $S_i\bar{x}(y)$ is the unique solution of (5.2), we have

$$S_i\bar{x}(y) = \bar{x}(S_i y), \qquad y \in \Omega.$$

We finally introduce, for $y \in \Omega$, the application

$$F: \Omega \to \operatorname{Im} Q, \qquad y \mapsto F(y) = QN\bar{x}(y). \tag{5.5}$$

The determining equations (2.7) can then be written as

$$F(y) = 0, \qquad y \in \Omega. \tag{5.6}$$

THEOREM 5.3. *We have, on Ω:*

$$S_i^* F = F S_i \qquad (i = 1,..., n). \tag{5.7}$$

The foregoing theorems show that all operators, appearing in the formulation of the alternative problem, remain symmetric on the condition that the projections P and Q, given in (H1), are chosen in a symmetric way, as explained in Section 4. We now show how this reduces the determining equations for some special solutions.

6. SYMMETRIC SOLUTIONS

As in the foregoing section, we assume (H1), (H2), and (H3). We consider, in the Banach spaces X, resp. Z, the continuous operators:

$$\mathcal{S} = \frac{1}{n}\sum_i S_i \qquad \text{resp.} \qquad \mathcal{S}^* = \frac{1}{n}\sum_i S_i^*. \tag{6.1}$$

LEMMA 6.1. *$\mathcal{S}$ and $\mathcal{S}^*$ are projections, respectively, in X and Z.*

Proof. $\mathcal{S}^2 = \mathcal{S}$ follows from a direct calculation, using the rearrangement theorem for group elements.

An analogous argument shows the following.

LEMMA 6.2. *We have for all $i = 1,..., n$:*

$$\begin{aligned} &\text{(i)} \quad \mathcal{S}S_i = S_i\mathcal{S} = \mathcal{S}; \\ &\text{(ii)} \quad \mathcal{S}^*S_i^* = S_i^*\mathcal{S}^* = \mathcal{S}^*. \end{aligned} \tag{6.2}$$

LEMMA 6.3.

$$\operatorname{Im} \mathcal{S} = \{x \in X : x = S_i x, i = 1,\dots, n\} = X_S . \tag{6.3}$$

$$\operatorname{Im} \mathcal{S}^* = \{z \in Z : z = S_i^* z, i = 1,\dots, n\} = Z_S . \tag{6.4}$$

Proof. We have from Lemma 6.2:

$$x = S_i x \qquad (i = 1,\dots, n) \Leftrightarrow x = \mathcal{S} x.$$

THEOREM 6.1. *Let $y \in \Omega \cap \operatorname{Im} \mathcal{S}$. Then:*

(i) $\bar{x}(y) = \mathcal{S}\bar{x}(y)$ *and* $\bar{x}(y) \in \operatorname{dom} L \cap \operatorname{Im} \mathcal{S}$, (6.5)

(ii) $F(y) = \mathcal{S}^* F(y)$ *and* $F(y) \in \operatorname{Im} Q \cap \operatorname{Im} \mathcal{S}^*$. (6.6)

Proof. We have from Theorems 5.2 and 5.3:

$$y \in \Omega, \qquad y = S_i y \Rightarrow S_i \bar{x}(y) = \bar{x}(S_i y) = \bar{x}(y),$$
$$y \in \Omega, \qquad y = S_i y \Rightarrow S_i^* F(y) = F(S_i y) = F(y).$$

Theorem 6.1 means that for a symmetric $y \in \Omega$, the determining equation can be written as

$$\mathcal{S}^* F(y) = 0, \qquad y \in \Omega \cap \operatorname{Im} \mathcal{S} \tag{6.7}$$

that is an equation in the subspace $\operatorname{Im} Q \cap \operatorname{Im} \mathcal{S}^*$ of $\operatorname{Im} Q$. If (6.7) has a solution, the corresponding solution of (2.1) will also be symmetric. The foregoing results can be formulated as follows.

THEOREM 6.2. *Assume* (H1), (H2), *and* (H3). *Let P and Q in* (H1) *be symmetrized.*

Let $x^* \in \operatorname{dom} L$ be a solution of (2.1), *and put $y^* = Px^*$. Suppose $y^* \in \Omega$ and $y^* = \mathcal{S} y^*$. Then:*

(i) $x^* = \bar{x}(y^*)$; (6.8)

(ii) $x^* = \mathcal{S} x^*$; (6.9)

(iii) y^* *satisfies:* $F(y^*) = \mathcal{S}^* F(y^*) = 0$. (6.10)

Inversely, let $y^* \in \Omega \cap \operatorname{Im} \mathcal{S}$ be a solution of

$$\mathcal{S}^* F(y) = 0. \tag{6.11}$$

Then

$$x^* = \bar{x}(y^*) = \mathcal{S}\bar{x}(y^*) \tag{6.12}$$

is a (*symmetric*) *solution of* (2.1).

We can conclude that, when taking $y \in \Omega$ in a symmetric way (that is: invariant for the S_i), the corresponding solution of (5.1) will also be symmetric, and the determining equations reduce to their symmetric part (6.11). This means in practice a reduction of the space of the unknown y (from $\ker L$ to $\ker L \cap \operatorname{Im} \mathscr{S}$), and a corresponding reduction of the set of determining equations: the left members of (6.11) belong to $\operatorname{Im} Q \cap \operatorname{Im} \mathscr{S}^*$, while for the general problem (2.7), they belong to $\operatorname{Im} Q$.

It is clear that analogous results can be obtained for every subgroup $\mathscr{G}'$ of $\mathscr{G}$, and the corresponding subgroup $\mathscr{G}^{*\prime}$ of $\mathscr{G}^*$. One then looks for solutions remaining invariant for the symmetry operators of the subgroup. When taking $y \in \Omega$ in some special subspaces of $\ker L$, one can even look for those symmetry operators that leave the subspace invariant: these form a subgroup of $\mathscr{G}$, for which the foregoing theory can be applied. When y belongs to the subspace, the corresponding determining equations will also reduce in the way described above.

7. Applications

We first show that the results found by Hale [1, 2] for systems which have the property E, form a special case of our results. To do so, we consider the Banach spaces:

$$X = Z = C_T = \{x\colon R \to R^n,\ x \text{ is continuous and } T\text{-periodic}\}.$$

We are looking for T-periodic solutions of the T-periodic equation

$$\dot{x} = A(t)\, x + f(t, x). \tag{7.1}$$

Equation (7.1) is said to have the property E when there exists a constant, symmetric, and orthogonal matrix S, such that:

$$SA(-t) = -A(t)\, S, \qquad Sf(-t, Sx) = -f(t, x) \qquad \forall t \in R, \quad \forall x \in R^n. \tag{7.2}$$

We define:

$$L\colon C_T{}^1 \to C_T, \qquad x(t) \mapsto \left[\frac{d}{dt} - A(t)\right] x(t) \tag{7.3}$$

and

$$N\colon C_T \to C_T, \qquad x(t) \mapsto f(t, x(t)). \tag{7.4}$$

Then (7.1) can be written in the form $Lx = Nx$. Now we introduce the symmetry operators:

$$\begin{aligned} S_0 &= I_{C_T}; \qquad S_1\colon C_T \to C_T, \qquad x(t) \mapsto Sx(-t), \\ S_0{}^* &= I_{C_T}; \qquad S_1{}^*\colon C_T \to C_T, \qquad x(t) \mapsto -Sx(-t). \end{aligned} \tag{7.5}$$

Then $\mathscr{G} = \{S_0, S_1\}$ and $\mathscr{G}^* = \{S_0^*, S_1^*\}$ form two homomorphous groups in C_T. It is easy to check that, because of (7.2):

$$(LS_1)\, x = (S_1^* L)\, x \qquad \forall x \in C_T, \tag{7.6}$$

while also:

$$\begin{aligned}[NS_1 x]\,(t) = f(t, Sx(-t)) &= -Sf(-t, x(-t)) \\ &= [S_1^* Nx]\,(t).\end{aligned} \tag{7.7}$$

So Hypotheses (H2) and (H3) are satisfied.

Defining in C_T the inner product:

$$\langle x, y\rangle = \frac{1}{T}\int_0^T x^*(t)\, y(t)\, dt \tag{7.8}$$

it is easy to see that S_1 and S_1^* are unitary and Hermitian.

As $\ker L$ is finite dimensional, one can take an orthonormal basis $\{\phi_i: 1 \leqslant i \leqslant s\}$ in $\ker L$, and define:

$$Px = \sum_i \langle \phi_i, x\rangle\, \phi_i. \tag{7.9}$$

Then:

$$S_1^{-1} P S_1 x = S_1 P S_1 x = \sum_i \langle \phi_i, S_1 x\rangle\, S_1\phi_i = \sum_i \langle S_1\phi_i, x\rangle\, S_1\phi_i = Px \tag{7.10}$$

as $\{S_1\phi_i: 1 \leqslant i \leqslant s\}$ also forms an orthonormal basis of $\ker L$. Thus:

$$PS_i = S_i P \qquad (i = 0, 1). \tag{7.11}$$

So, P is a symmetric projection on $\ker L$. When taking in an analogous way Q as an orthogonal projection on $\ker L^*$, we will find that Q commutes with S_i^*, $i = 0, 1$. This is exactly the way in which P and Q from (H1) are always chosen for the equation considered here. Then the symmetry conditions on P and Q, needed for the application of the foregoing theory, are automatically fulfilled.

All further results obtained by Hale are then simple consequences of the theorems of Sections 5 and 6. So the theory given here forms a generalization of the property E of Hale.

A slight generalization of the property E was given by Mawhin [4]. This author assumed the existence of a symmetric, orthogonal matrix S, and real numbers ϵ and τ, with $\epsilon^2 = 1$, $(1 + \epsilon)\,\tau = mT$, such that:

$$A(\epsilon t + \tau)\, S = SA(t), \qquad Sf(\epsilon t + \tau, Sx) = \epsilon f(t, x). \tag{7.12}$$

When taking

$$\begin{aligned} S_1&: C_T \to C_T, \qquad x(t) \mapsto Sx(\epsilon t + \tau), \\ S_1^*&: C_T \to C_T, \qquad x(t) \mapsto \epsilon Sx(\epsilon t + \tau) \end{aligned} \tag{7.13}$$

in (7.5), it is clear that the theory applies again.

Another simple example is obtained when one looks for nT-periodic solutions of the T-periodic equation (7.1). One then has to work in the Banach space $X = Z = C_{nT}$, and, in that space, the equation (7.1) will be symmetric for the group $\mathscr{G} = \mathscr{G}^* = \{S_m: m = 1,..., n\}$, with:

$$S_m: C_{nT} \to C_{nT}, \qquad x(t) \mapsto x(t + mT) \qquad (m = 1,..., n). \tag{7.14}$$

It is easy to verify that S_m is unitary for the inner product:

$$\langle x, y\rangle = \frac{1}{nT} \int_0^{nT} x^*(t)\, y(t)\, dt \tag{7.15}$$

in C_{nT}. Taking for $P^{(n)}$ and $Q^{(n)}$ orthogonal projections on $(\ker L)_n \subset C_{nT}$ (that is, on the space of nT-periodic solutions of $Lx = 0$), they will commute with S_m.

Further:

$$\mathscr{S}: x(t) \mapsto \frac{1}{n} \sum_{m=1}^{n} x(t + mT) \tag{7.16}$$

projects C_{nT} on C_T. Now consider the determining equations

$$F^{(n)}(y^{(n)}) = 0, \qquad y^{(n)} \in \Omega^{(n)} \subset (\ker L)_n. \tag{7.17}$$

Through application of the theory we then learn that, when taking

$$y^{(n)} = y \in \Omega^{(n)} \cap C_T = \Omega,$$

(7.17) reduces to the determining equations for the T-periodic solutions of (7.1). Although this is what one should expect, it must be remarked that this only happens when the projections $P^{(n)}$ and $Q^{(n)}$ in C_{nT} are taken in a symmetric way.

More substantially, the theory is applicable to the problem of finding periodic solutions for nonlinear vibration systems, when some spatial symmetry is present. The above theory, and especially the remark at the end of Section 6, can then be very helpful in finding solutions which have some symmetry properties. We hope to give some more detailed applications in this sense in a future paper.

Remark. The foregoing results show some interesting parallelisms with the results of Lewis in his paper "Autosynartetic Solutions of Differential Equations" [5]. Lewis shows how one obtains an alternative problem and a corresponding bifurcation equation for so-called autosynartetic solutions, that is, solutions remaining invariant for a certain transformation, which also leaves the given differential equation invariant. The existence of autosynartetic first integrals further reduces the bifurcation equation.

Here we started with a given alternative problem, having some supplementary symmetry, and showed how the bifurcation equation reduces for solutions remaining invariant for this symmetry group. Such solutions can in some sense be seen as autosynartetic solutions; that means solutions remaining invariant under some transformations, that also leave the given equation invariant.

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